

# Higher-dimensional geometric $\sigma$ -models

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**Abstract.** Geometric  $\sigma$ -models have been defined as purely geometric theories of scalar fields coupled to gravity. By construction, these theories possess arbitrarily chosen vacuum solutions. Using this fact, one can build a Kaluza–Klein geometric  $\sigma$ -model by specifying the vacuum metric of the form  $M^4 \times B^d$ . The obtained higher dimensional theory has vanishing cosmological constant but fails to give massless gauge fields after the dimensional reduction. In this paper, a modified geometric  $\sigma$ -model is suggested, which solves the above problem.

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## 1. Introduction

The cosmological constant problem of Kaluza–Klein theories [1] whose internal manifold is not Ricci-flat is a longstanding one. In the conventional Kaluza–Klein treatment, the internal manifold is chosen in such a way that its isometries define internal symmetries of the theory. At the same time, a ground state in the form of the direct product of the 4-dimensional flat spacetime with a compact, nonflat internal space does not satisfy classical Einstein–Hilbert equations of motion. The attempt to solve this problem by adding a cosmological term has failed. Indeed, to reproduce some known gauge couplings, the cosmological constant is constrained to be of the order of the Planck mass squared, which strongly disagrees with the observed universe.

Among a variety of existing approaches to this problem, we shall focus our attention on those which use matter fields to trigger spontaneous compactification. At the same time, we do not want to lose the geometric character of our theory. How can we reconcile these two requirements? Notice, in this respect, that so called geometric  $\sigma$ -models have been defined [2] as purely geometric theories of scalar fields coupled to gravity. By construction, these scalar fields originate from the coordinates of the spacetime, and, as a consequence, can be gauged away. In the context of higher dimensional theories, such an approach has already been used in literature. The authors of references [3] and [4] have employed scalar fields in the form of a nonlinear  $\sigma$ -model to trigger the compactification. It is not difficult to see that their model is a particular example of a geometric  $\sigma$ -model. It turns out, however, that, although solved the cosmological constant problem, it failed to give massless gauge fields. We shall try to modify the model of [3] and [4] in the spirit of [2], and reconcile the masslessness of the gauge fields with the zero value of cosmological constant. In the course of our analysis, it will become obvious that more than one model of the kind can be defined.

The lay-out of the paper is as follows. In section 1, we shall analyse the model suggested in [3] and [4] from the point of view of geometric  $\sigma$ -models. We

shall readily use the gauge freedom, and fix all the scalar fields in the theory thereby reducing it to the purely geometric, non-covariant equations of motion of the form  $R_{MN} = \overset{\circ}{R}_{MN}$ . The function  $\overset{\circ}{R}_{MN}$  on the right-hand side is the Ricci tensor of the Kaluza–Klein vacuum  $M^4 \times B^d$ . The linearized version of the theory turns out to be easily averaged over the internal coordinates. The resulting effective 4-dimensional equations of motion are then shown to necessarily contain massive terms in the sector of gauge fields. A careful analysis of the problem suggests a simple modification of this model. In section 3, we shall see that the Yang–Mills sector of the effective 4-dimensional equations of motion obtained by averaging the equations  $R^{MN} = \overset{\circ}{R}^{MN}$  contains no massive fields. The fact that the new model uses Ricci tensors with upper indices turns out to be crucial. However, it makes it difficult to construct the corresponding Lagrangian. This is why we suggest another modification of the model in section 4. By adding terms proportional to  $(G_{MN} - \overset{\circ}{G}_{MN})$  to our equations of motion, we shall certainly not lose the good property of our model to have vanishing cosmological constant. Indeed, such equations possess the vacuum solution  $G_{MN} = \overset{\circ}{G}_{MN}$ , and we choose the metric  $\overset{\circ}{G}_{MN}$  to be of the Kaluza–Klein type  $M^4 \times B^d$ . We shall be able to demonstrate that, after the dimensional reduction, our higher dimensional Lagrangian gives the standard Einstein, Yang–Mills and Klein–Gordon sectors. The scalar excitations of the internal manifold turn out to be all massive, with masses of the order of the Planck mass. The analysis is confined within the linear approximation of the theory.

In section 4, we shall covariantize our model. By employing a set of  $4 + d$  scalar fields, a generally covariant  $\sigma$ -model of a non-standard type is obtained. The Lagrangian turns out to be a non-polynomial function of the scalar field derivatives. We shall still be able to bring it to a polynomial form by introducing a set of auxiliary fields. The obtained theory retains a purely geometric character since all the fields except  $G_{MN}$  are either auxiliary or gauge degrees of freedom. A brief comparison of the new model with conventional nonlinear  $\sigma$ -models points out some conspicuous differences.

Section 5 is devoted to concluding remarks.

## 2. Compactification induced by scalars

The model the authors of references [3] and [4] discuss consists of Einstein gravity in  $4 + d$  dimensions coupled to a nonlinear  $\sigma$ -model:

$$I = -\kappa^2 \int d^{4+d} X \sqrt{-G} \left[ R - F_{ij}(\Omega) \Omega^i_{,M} \Omega^j_{,N} G^{MN} \right] . \quad (1)$$

The scalar fields  $\Omega^i$ ,  $i = 1, 2, \dots, d$ , are thought of as coordinates of a  $d$ -dimensional compact Riemannian manifold  $B^d$  with Ricci tensor  $F_{ij}(\Omega)$ , while the coordinates  $X^M \equiv (x^\mu, y^m)$  parametrize a  $(4 + d)$ -dimensional spacetime with metric  $G_{MN}$ . The indices run as follows:

$$M, N = 0, 1, \dots, 3 + d$$

$$\mu, \nu = 0, 1, \dots, 3$$

$$m, n = 1, 2, \dots, d .$$

Notice that the number of scalar fields equals the number of compact dimensions of the spacetime. The equations of motion for this theory possess a vacuum solution of the form

$$G_{MN} = \overset{o}{G}_{MN} \quad \Omega^m = y^m \quad (2)$$

where

$$\overset{o}{G}_{MN} \equiv \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \phi_{mn}(y) \end{pmatrix} \quad (3)$$

and  $\phi_{mn}(y)$  stands for the metric of  $B^d$ . The scalar sector of the solution (2) is obviously topologically nontrivial since it is described by a degree one mapping from  $B^d$  to  $B^d$ . At the same time, the metric (3) has the form of the direct product of the 4-dimensional Minkowski spacetime with a compact internal space, as desired. If we restrict our attention to the physics of small excitations of this vacuum, we can always choose the spacetime coordinates to fix  $\Omega^m = y^m$ . Then, the action functional (1) reduces to

$$I = -\kappa^2 \int d^{4+d} X \sqrt{-G} \left( R_{MN} - \overset{o}{R}_{MN} \right) G^{MN} . \quad (4)$$

where  $\overset{o}{R}_{MN}$  stands for the Ricci tensor of the vacuum metric (3). This is a purely

geometric but non-covariant theory whose physical content is fully contained in the equations of motion

$$R_{MN} = \overset{o}{R}_{MN} . \quad (5)$$

Comparing it with the results of [2], we can see that the above theory is a particular example of a geometric  $\sigma$ -model based on the vacuum metric (3). The covariantization of the equations (5) in the spirit of [2] introduces  $4 + d$  scalar fields in the form of a nonlinear  $\sigma$ -model. It is owing to the special form of  $\overset{o}{G}_{MN}$  as given by (3) that only  $d$  out of  $4 + d$  scalar fields survive.

As mentioned in the introduction, the model (1) solves the cosmological constant problem but fails to give massless Yang–Mills fields after the dimensional reduction. When rewritten as (5), the theory automatically takes care of the cosmological term. Indeed, the metric  $\overset{o}{G}_{MN}$  is by definition a solution to the equations of motion (5). To analyse the spectrum of the corresponding effective 4-dimensional theory, we shall use the standard  $4 + d$  decomposition [5] of the metric  $G_{MN}$ :

$$G_{MN} \equiv \begin{pmatrix} g_{\mu\nu} + B_\mu^k B_\nu^l u_{kl} & B_\mu^k u_{kn} \\ B_\nu^k u_{km} & u_{mn} \end{pmatrix} . \quad (6)$$

For the perturbations of the vacuum  $\overset{o}{G}_{MN}$  we adopt the notation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad u_{mn} = \phi_{mn} + \varphi_{mn} . \quad (7)$$

Substituting these expressions into (5) we find

$$\begin{aligned} R^\mu{}_\nu &= 0 \\ R^\mu{}_n &= -\overset{o}{R}_{nl} B^{l\mu} + O(2) \\ R^m{}_n &= \overset{o}{R}^m{}_n - \overset{o}{R}_{nl} \varphi^{ml} + O(2) \end{aligned} \quad (8)$$

where  $\overset{o}{R}_{mn}$  is the  $d$ -dimensional Ricci tensor of the vacuum metric  $\phi_{mn}$ . For our purposes, the mixed components  $R^M{}_N$  of the Ricci tensor turn out to be more convenient. Indeed, it is the mixed components which, after the decomposition (6) is employed, give the standard Einstein and Yang–Mills terms. The expressions on the right-hand

side of (8) then measure the deviation of our theory from the standard case. In particular, we shall see that the term  $\overset{o}{R}_{nl}B^{l\mu}$  is responsible for the appearance of massive gauge fields.

To obtain effective 4-dimensional equations of motion, we shall average the equations (8) over the internal coordinates. The average of a d-scalar  $S$ , as defined by

$$\langle S \rangle \equiv \frac{\int d^d y \sqrt{-u} S}{\int d^d y \sqrt{-u}} = \frac{\int d^d y \sqrt{-\phi} S}{\int d^d y \sqrt{-\phi}} + O(2) \quad (9)$$

is also a d-scalar. However, a simple definition of the kind for d-vectors and d-tensors does not exist. This is why we have to project the equations (8) on an appropriately chosen basis in  $B^d$  before we use (9) to average them. (This kind of dimensional reduction has already been used in [6] to obtain effective 4-dimensional equations of motion out of dimensionally continued Euler forms.) As is customary, let us suppose that  $B^d$  is a homogeneous space with  $m$  Killing vectors  $K_a^l(y)$ ,  $a = 1, \dots, m$ , which form a (generally overcomplete) basis in  $B^d$ . Using the decomposition

$$B_\nu^m = K_a^m A_\nu^a \quad \varphi_{mn} = K_{am} K_{bn} \varphi^{ab}$$

and projecting the vector and tensor equations (8) on the Killing basis, we obtain an equivalent set of d-scalar field equations. Although basically linear, the averaged equations will contain terms of the type  $\langle S_1 S_2 \rangle$  owing to the presence of  $y$ -dependent coefficients. The average  $\langle S_1 S_2 \rangle$  cannot generally be expressed in terms of  $\langle S_1 \rangle$  and  $\langle S_2 \rangle$ . However, our internal manifold is of the Planckian size, and it is not unreasonable to restrict our analysis to solutions which slowly vary in  $y$  direction. In that case, the product of averages  $\langle S_1 \rangle \langle S_2 \rangle$  becomes the leading term in the decomposition

$$\langle S_1 S_2 \rangle = \langle S_1 \rangle \langle S_2 \rangle + \Delta_{12}$$

so that  $\Delta_{12}$  can be regarded as a small correction. Using this fact and the fact that averages of covariant d-divergences vanish, we obtain the following effective

4-equations:

$$\begin{aligned}
\bar{\mathcal{R}}_{\mu\nu} + \frac{1}{2} \bar{\varphi}_{,\mu\nu} &= \emptyset \\
\gamma_{ab} \partial_\nu \bar{F}^{b\mu\nu} + 2 m_{ab} \bar{A}^{b\mu} &= \emptyset \\
\sigma_{abcd} \square \bar{\varphi}^{cd} + \mu_{abcd} \bar{\varphi}^{cd} &= \emptyset .
\end{aligned} \tag{10}$$

Here,  $\mathcal{R}_{\mu\nu}$  is the Ricci tensor of the 4-metric  $g_{\mu\nu}$ ,  $\square$  is the corresponding d'Alembertian, and  $F_{\mu\nu}^a \equiv A_{\mu,\nu}^a - A_{\nu,\mu}^a + O(2)$  is the gauge field strength for the gauge fields  $A_\mu^a$ . The bar over a quantity denotes its expectation value as defined by (9), and  $\emptyset \equiv \Delta + O(2)$ . The coefficients in (10) are vacuum expectation values of products of the Killing vectors and their covariant derivatives. For example,

$$\begin{aligned}
\gamma_{ab} &\equiv \langle K_a^m K_{bm} \rangle & m_{ab} &\equiv \langle K_a^m K_b^n \overset{\circ}{R}_{mn} \rangle \\
\sigma_{abcd} &\equiv \langle K_a^m K_b^n K_{cm} K_{dn} \rangle + a \leftrightarrow b .
\end{aligned}$$

We see that the fields  $\bar{A}_\mu^a$  are generally massive with masses proportional to the curvature of the internal manifold  $B^d$ . So are the scalar excitations  $\bar{\varphi}^{ab}$ , with the exception of  $\bar{\varphi} \equiv \gamma_{ab} \bar{\varphi}^{ab} = \langle \varphi_m^m \rangle + \Delta$  which turns out to be massless. We can use that fact to rescale the metric  $\bar{g}_{\mu\nu}$  according to  $\tilde{g}_{\mu\nu} \equiv (1 + \bar{\varphi}/2) \bar{g}_{\mu\nu}$ , whereby the first equation (10) takes the standard Einstein form  $\tilde{\mathcal{R}}_{\mu\nu} = \emptyset$ .

### 3. Massless gauge fields

By analysing the equations (8), one finds that the term  $\overset{\circ}{R}_{nl} B^{l\mu}$  on the right-hand side makes the gauge fields massive. The simplest way to get rid of it is to postulate the equations of motion of the form  $R_N^M = \overset{\circ}{R}_N^M$ . These, however, are not symmetric, and can only be used within the vielbein formalism. For this reason, we shall concentrate our attention on the symmetric field equations of the form

$$R^{MN} = \overset{\circ}{R}^{MN} . \tag{11}$$

It is not difficult to show that the corresponding theory contains no massive gauge fields. Indeed, by rewriting the equations (11) in terms of the mixed components of

the Ricci tensor, we find

$$\begin{aligned}
R^\mu{}_\nu &= 0 \\
R^\mu{}_n &= 0 \\
R^m{}_n &= \overset{o}{R}^m{}_n + \overset{o}{R}^{ml} \varphi_{nl} .
\end{aligned} \tag{12}$$

The critical term on the right-hand side of the second equation (12) is missing! The effective 4-dimensional equations of motion are obtained by the exact procedure described in section 2. The result is

$$\begin{aligned}
\bar{\mathcal{R}}_{\mu\nu} + \frac{1}{2} \bar{\varphi}_{,\mu\nu} &= 0 \\
\partial_\nu \bar{F}_a{}^{\mu\nu} &= 0 \\
\sigma_{abcd} \square \bar{\varphi}^{cd} + \mu'_{abcd} \bar{\varphi}^{cd} &= 0
\end{aligned} \tag{13}$$

where the group metric  $\gamma_{ab}$  is used to raise and lower the group indices. Comparing it to (10), we see that there are no massive terms in the Yang–Mills sector. In addition, the formerly massless scalar field  $\bar{\varphi}$  acquires mass of the order of the Planck mass. In particular, if we choose our  $B^d$  to be an Einstein manifold, say  $\overset{o}{R}_{mn} = \lambda \phi_{mn}$ , we shall find  $(\square - 4\lambda)\bar{\varphi} = 0$ . As a consequence of  $\square \bar{\varphi} \neq 0$ , the local rescalings of the metric  $\bar{g}_{\mu\nu}$  cannot bring the first equation (13) into the standard Einstein form. Still, it is possible to fix the gauge  $\partial^\nu \psi_{\mu\nu} = \frac{1}{2} \bar{\varphi}_{,\mu}$  in the linearized theory ( $\psi_{\mu\nu} \equiv \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{h}^\lambda{}_\lambda$ ) thereby reducing  $\bar{\mathcal{R}}_{\mu\nu} + \frac{1}{2} \bar{\varphi}_{,\mu\nu} = 0$  to  $\square \bar{h}_{\mu\nu} = 0$ , as is customary. In this respect, notice that, although the equations (11) are basically non-covariant, they still possess a partial gauge symmetry as a consequence of our special choice of  $B^d$ . Indeed, it is not difficult to check that the coordinate transformations

$$x^{\mu'} = x^{\mu'}(x) \quad y^{m'} = y^m + \epsilon^a(x) K_a^m(y)$$

do not change the form of the equations of motion (11).

Before we covariantize the non-covariant field equations (11), we would like to define the corresponding action functional. It turns out, however, that no obvious generalization of (4) exists. In the next section, we shall suggest a Lagrangian whose equations of motion differ from (11) but retain all their good features.



#### 4. Lagrangian

The geometric  $\sigma$ -model approach to the cosmological constant problem does not uniquely single out the equations of motion in the form of (5) or (11). One can always add terms proportional to  $(G_{MN} - \overset{\circ}{G}_{MN})$  without losing the vacuum solution  $G_{MN} = \overset{\circ}{G}_{MN}$ . We shall use this freedom to modify the equations (11) in a way which will allow for a simple construction of the corresponding Lagrangian. At the same time, we have to carefully choose this correction in order not to lose the needed masslessness of the Yang–Mills sector. A simple analysis along these lines takes us to the following non-covariant action functional

$$I = -\kappa^2 \int d^{4+d}X \sqrt{-G} \left[ R - \overset{\circ}{R} + \overset{\circ}{R}^{MN} \left( G_{MN} - \overset{\circ}{G}_{MN} \right) \right] . \quad (14)$$

Varying it with respect to  $G_{MN}$  gives the equations of motion of the form

$$R^{MN} = \overset{\circ}{R}^{MN} - \frac{2}{2+d} G^{MN} \overset{\circ}{R}^{LR} \left( G_{LR} - \overset{\circ}{G}_{LR} \right) . \quad (15)$$

As we can see, the correction to (11) is indeed proportional to  $(G_{MN} - \overset{\circ}{G}_{MN})$ . The Yang–Mills sector of the theory is best analysed if we rewrite (15) using mixed components of the Ricci tensor. Then, the equations of motion read

$$\begin{aligned} R^\mu_\nu &= -\frac{2}{2+d} \delta^\mu_\nu \overset{\circ}{R}^{ij} \varphi_{ij} \\ R^\mu_n &= 0 \\ R^m_n &= \overset{\circ}{R}^m_n + \overset{\circ}{R}^{ml} \varphi_{nl} - \frac{2}{2+d} \delta^m_n \overset{\circ}{R}^{ij} \varphi_{ij} . \end{aligned} \quad (16)$$

As in (12), the crucial term on the right-hand side of the second equation (16) is missing. The effective 4-dimensional equations of motion are obtained using the averaging procedure of section 2. To simplify the analysis, we shall choose our internal space in the form of an Einstein manifold

$$\overset{\circ}{R}_{mn} = \lambda \phi_{mn}$$

with  $\lambda < 0$  in accordance with the adopted conventions ( $R^M_{NL,R} = \Gamma^M_{NL,R} - \dots$ ,

$diag(G_{MN}) = (-, +, \dots, +)$ . Then, the averaged equations become

$$\begin{aligned}\bar{\mathcal{R}}_{\mu\nu} + \frac{1}{2} \bar{\varphi}_{,\mu\nu} + \frac{2\lambda}{d+2} \eta_{\mu\nu} \bar{\varphi} &= 0 \\ \partial_\nu \bar{F}_a{}^{\mu\nu} &= 0 \\ \sigma_{abcd} \square \bar{\varphi}^{cd} + \mu''_{abcd} \bar{\varphi}^{cd} &= 0.\end{aligned}\tag{17}$$

The gauge fields  $\bar{A}_\mu^a$  are obviously massless, but the scalar excitations  $\bar{\varphi}^{ab}$  have masses of the order of the Planck mass. In particular, the scalar field  $\bar{\varphi}$ , appearing in the first equation (17), satisfies

$$\left( \square - \frac{8\lambda}{d+2} \right) \bar{\varphi} = 0.\tag{18}$$

We see that the conventional choice  $\lambda < 0$  ensures the correct sign for the mass term in (18). Moreover, as opposed to the case of section 3, the equation (18) makes it possible to rescale the metric  $\bar{g}_{\mu\nu}$  according to

$$\tilde{g}_{\mu\nu} \equiv \left( 1 + \frac{\bar{\varphi}}{2} \right) \bar{g}_{\mu\nu} + O(2)$$

thereby bringing the first equation (17) into the standard Einstein form

$$\tilde{\mathcal{R}}_{\mu\nu} = 0.$$

The masses  $\mu''_{abcd}$ , as well as the coefficients  $\sigma_{abcd}$  and  $\gamma_{ab}$ , are defined as vacuum expectation values of products of the Killing vectors and their covariant derivatives. They are constant tensors of the isometry group of the internal manifold  $B^d$ . In the case of  $B^d = S^2$ , for example, one finds

$$\begin{aligned}\gamma_{ab} &= \frac{2}{3} \delta_{ab} \\ \sigma_{abcd} &= \frac{2}{15} \delta_{ab} \delta_{cd} + \frac{7}{15} (\delta_{ac} \delta_{bd} + \delta_{bc} \delta_{ad}).\end{aligned}$$

The  $SO(3)$  tensor  $\sigma_{abcd}$  has the inverse defined through  $(\sigma^{-1})^{abcd} \sigma_{cdef} \equiv \delta_{(e}^a \delta_{f)}^b$ . As a consequence, all the scalar fields  $\bar{\varphi}^{ab}$  survive as independent degrees of freedom in this theory. This is an improvement as compared to [6] where the cosmological constant problem has been solved at the expense of losing the kinetic terms of some

scalar excitations. The mass matrix  $\mu''_{abcd}$ , being a constant  $SO(3)$  tensor itself, has the same structure as  $\sigma_{abcd}$ , but requires a lengthier calculation.

## 5. Covariantization

To covariantize the theory given by the action functional (14), we shall follow the ideas of reference [2]. Like there, we shall use a new set of coordinates,  $\Omega^A = \Omega^A(X)$ ,  $A = 0, 1, \dots, 3 + d$ , to fix the vacuum quantities of our model. Then, the covariantization is achieved through the substitution

$$\begin{aligned}\overset{\circ}{R}{}^{MN}(X) &\rightarrow \overset{\circ}{R}{}^{AB}(\Omega) \frac{\partial X^M}{\partial \Omega^A} \frac{\partial X^N}{\partial \Omega^B} \\ \overset{\circ}{R}(X) &\rightarrow \overset{\circ}{R}(\Omega)\end{aligned}$$

in the equations of motion (15) or, equivalently, Lagrangian (14). This gives

$$I = -\kappa^2 \int d^{4+d}X \sqrt{-G} \left[ R + F^{AB}(\Omega) \frac{\partial X^M}{\partial \Omega^A} \frac{\partial X^N}{\partial \Omega^B} G_{MN} - V(\Omega) \right] \quad (19)$$

where the target metric  $F^{AB}(\Omega)$  and the potential  $V(\Omega)$  are defined as

$$F^{AB}(\Omega) \equiv \overset{\circ}{R}{}^{AB}(\Omega)$$

$$V(\Omega) \equiv 2 \overset{\circ}{R}(\Omega) .$$

The identification of the new coordinates with the old ones,  $\Omega^A = X^A$ , takes us back to the non-covariant theory. The higher dimensional Lagrangian (19) looks like a  $\sigma$ -model coupled to gravity, but is certainly not of a standard type. The derivatives of the scalar fields  $\Omega^A$  appear non-polynomially in the action. We can bring it to a polynomial form by introducing a set of auxiliary fields. In particular, we need  $4 + d$  vector fields  $b^A_M$  subject to the equations of motion  $b^A_M = \Omega^A_{,M}$ . The easiest way to achieve this is to postulate the action functional

$$I = -\kappa^2 \int d^{4+d}X \sqrt{-G} \left[ R + F^{AB}(\Omega) h^M_A h^M_B + \lambda^M_A (b^A_M - \Omega^A_{,M}) - V(\Omega) \right] \quad (20)$$

with  $h^M_A$  the inverse of  $b^A_M$ , and Lagrange multipliers  $\lambda^M_A$ . That it is indeed equivalent

to (14) is shown by inspecting the equations of motion. One finds

$$b^A_M = \Omega^A_{,M} \quad (21a)$$

$$\lambda^M_A = 2 F^{BC}(\Omega) h^L_B h^M_C h_{LA} \quad (21b)$$

$$R^{MN} = F^{AB}(\Omega) h^M_A h^N_B - \frac{2}{d+2} G^{MN} \left[ F^{AB}(\Omega) h^L_A h_{LB} - \frac{1}{2} V(\Omega) \right] \quad (21c)$$

$$\frac{\partial V}{\partial \Omega^A} = \frac{\partial F^{BC}}{\partial \Omega^A} h^M_B h_{MC} - \frac{1}{2} G_{MN} G^{MN}_{,L} \lambda^L_A + \lambda^M_{A,M} . \quad (21d)$$

The auxiliary fields  $b^A_M$  and  $\lambda^M_A$  are fully expressed in terms of  $G_{MN}$  and  $\Omega^A$ , and carry no degrees of freedom. The equation (21d), obtained by varying the action (20) with respect to  $\Omega^A$ , is not an independent equation of motion. It is easily shown to follow from the Bianchi identities  $(R^M_N - \frac{1}{2} \delta^M_N R)_{;M} \equiv 0$  and (21a-c). It turns out then that the content of the theory is fully contained in (21c) with  $h^M_A = \partial X^M / \partial \Omega^A$ . After the spacetime coordinates are chosen to fix  $\Omega^A = X^A$ , the equations of motion boil down to (15), as expected.

The theory given by (20) or, equivalently, (19) differs in some aspects from the geometric  $\sigma$ -models of reference [2], and, in that respect, from the model of references [3] and [4]. First, the equations of motion (21) do not admit the topologically trivial solution  $\Omega^A = 0$  representing a non-geometric sector of the theory. Second, although our target metric  $F^{AB}(\Omega)$  has vanishing  $F^{\alpha B}$  ( $\alpha = 0, \dots, 3$ ) components, we still need all  $4+d$  fields  $\Omega^A$ . In ordinary geometric  $\sigma$ -models, the rank of the Ricci tensor  $\overset{\circ}{R}_{MN}$  determines the number of necessary scalar fields. This is why we needed only  $d$  out of  $4+d$  scalar fields in the model defined by (1). Here, we have to retain all the components  $\Omega^A$ , in particular  $\Omega^0$  whose vacuum value  $\Omega^0 = X^0$  is time dependent. Still, this is a pure coordinate time dependence which can easily be gauged away.

## 6. Concluding remarks

We have applied the ideas of geometric  $\sigma$ -models [2] to solve the cosmological constant problem of Kaluza-Klein theories. This kind of approach is not new in

literature. The authors of references [3] and [4] have demonstrated how scalar fields in the form of a  $\sigma$ -model can trigger spontaneous compactification. It turned out, however, that their model failed to give massless gauge fields after the dimensional reduction. We have rewritten this theory in terms of a geometric  $\sigma$ -model, thereby bringing it to a suggestive form. It was not difficult then to realize which kind of modification would reconcile the masslessness of the gauge fields with the zero value of the cosmological constant. In section 3, the modified theory has been proven to contain no massive gauge fields. The effective 4-dimensional theory has been obtained by averaging the linearized  $(4 + d)$ -dimensional equations of motion over the internal coordinates.

In search for the simple Lagrangian of the theory, we had to abandon the model of section 3, and look for another modification. We have found an action functional whose equations of motion differ from those of section 3 by the presence of a term proportional to  $(G_{MN} - \overset{o}{G}_{MN})$ , but retain all their good features. The linearized effective 4-dimensional equations of motion turned out to contain the standard Einstein, Yang–Mills and Klein–Gordon sectors. In addition, the scalar excitations of the internal manifold, in particular their zero mode, have been shown to have masses of the order of the Planck mass.

In section 5, we have covariantized our theory. A set of  $4 + d$  scalar fields has been introduced in a purely geometric manner. The generally covariant theory turned out to be of the form of a non-standard  $\sigma$ -model with non-polynomial dependence on the scalar field derivatives. We have demonstrated how the introduction of auxiliary fields brings it to a polynomial form. Compared to geometric  $\sigma$ -models of reference [2], and, in that respect, to the model of [3] and [4], our theory exhibits some differences. In particular, the number of scalar fields needed for the covariantization does not match the rank of the vacuum value of the Ricci tensor.

In the course of our analysis, it became obvious that the form of the dynamics was chosen from a variety of possibilities. To decide upon one, we have to study its physical implications. The first thing one should check is the general stability of the vacuum state. If the dynamics of the theory does not support the stable  $M^4 \times B^d$

vacuum configuration, it should be abandoned. If it does, we still have to compare the implications of the interacting theory with the known results. In search for a realistic theory of the kind, we could also further develop the idea of [2] to give fermions a pure geometric origin. In particular, it would be more in the spirit of geometric  $\sigma$ -models if we chose our vacuum metric in the form of a localized, particle-like field configuration which only asymptotically approaches  $M^4 \times B^d$ . The corresponding theory of the type considered in this paper might turn out to be more promising.

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